Axioms of Geometric Algebra

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1 Axioms for Geometric Algebra $\mathcal{R}_{p,q}$

1.1 Algebra over Field of Real Numbers

The set $\mathbb{R}$ of real numbers forms a field. For the addition of $a, b, c \in \mathbb{R}$ we have the properties of

$a + b = b + a$	commutativity
$(a + b) + c = a + (b + c)$	associativity
$a + 0 = a$	zero 0
$a + (-a) = 0$	opposite $-a$ of a.

For the multiplication of $a, b, c \in \mathbb{R}$ we have the properties of

$(a + b)c = bc + ac$	
$a(b + c) = ab + ac$	distributivity
$(ab)c = a(bc)$	associativity
$1a = a$	unity $1 \neq 0$
$aa^{-1} = 1$	inverse a^{-1} for $a \neq 0$.
$ab = ba$	commutativity.

Definition. An algebra over $\mathbb{R}$ is a linear space A over $\mathbb{R}$ together with a bilinear map (implying distributivity) $A \times A \to A, (a, b) \to ab$.

1.2 Definition (1) Using Quadratic Form

Let $V = (\mathbb{R}^n, \mathcal{Q})$ be a real vector space with a non-degenerate quadratic form $\mathcal{Q} : V \to \mathbb{R}$ of signature (p, q) with $p + q = n$. It is conventional to use the abbreviation $\mathbb{R}^{p,q}$. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{p,q}$ we associate with $\mathcal{Q}$ the symmetric bilinear form $\langle \mathbf{ab} \rangle = \frac{1}{2}[\mathcal{Q}(\mathbf{a} + \mathbf{b}) - \mathcal{Q}(\mathbf{a}) - \mathcal{Q}(\mathbf{b})]$.

Definition. An associative algebra over the field $\mathbb{R}$ with unity 1 is the *geometric algebra* $\mathbb{R}_{p,q}$ of the non-degenerate quadratic form $\mathcal{Q}$ (signature (p, q) and $p+q = n$) on $\mathbb{R}^n$, which contains copies of the field $\mathbb{R}$ and of the vector space $\mathbb{R}^{p,q}$ as distinct subspaces so that

(1) $\mathbf{a}^2 = \mathcal{Q}(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{R}^{p,q}$

(2) $\mathbb{R}^{p,q}$ generates $\mathbb{R}_{p,q}$ as an algebra over the field $\mathbb{R}$

(3) $\mathbb{R}_{p,q}$ is not generated by any proper subspace of $\mathbb{R}^{p,q}$.

Note the deliberate choice of unity 1! If $q = 0$ the second index is often omitted

$$\mathbb{R}_n = \mathbb{R}_{n,0}.$$

Using an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for $\mathbb{R}^{p,q}$, the condition (1) can be expressed as

$$\mathbf{e}_k^2 = 1, \quad 1 \le k \le p, \quad \mathbf{e}_k^2 = -1, \quad p < k \le n, \quad \mathbf{e}_k \mathbf{e}_l = -\mathbf{e}_l \mathbf{e}_k, \quad k < l.$$

(3) is only needed for signatures $p - q = 1 \bmod 4$ where $(\mathbf{e}_1 \mathbf{e}_2 \dots \mathbf{e}_n)^2 = 1$.

The basic multiplication rules for the basis vectors of an orthonormal basis can also be used to define a geometric algebra. This was the approach taken by Clifford himself in 1878 and 1882.

1.3 Definition (2) by Basic Multiplication Rules

Historically the first multiplication rule for vectors of the linear space $\mathbb{R}^n$ of importance for us is the bilinear outer (exterior) product of vectors. For a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ Grassmann introduced "bivectors"

$$\mathbf{e}_k \wedge \mathbf{e}_l = -\mathbf{e}_l \wedge \mathbf{e}_k, k \neq l$$

$$\mathbf{e}_k \wedge \mathbf{e}_l = 0, k = l.$$

The set of all bivectors $\{\mathbf{e}_k \wedge \mathbf{e}_l | k < l\}$ forms a basis of a new linear space $\bigwedge^2 \mathbb{R}^n$ of dimension $\begin{pmatrix} n \\ 2 \end{pmatrix}$. The name outer (or exterior) product stems from the fact, that it assigns to any pair of vectors of the original space $\mathbb{R}^n$ an element of a different vector space $\bigwedge^2 \mathbb{R}^n$.

Grassmann then defines multivectors. To any r-tuple $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ of vectors $\mathbf{v}_k \in \mathbb{R}^n$ he assignes the multivector of grade r (or rank r, German: Stufe r)

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_r.$$

$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_r$ is set zero if the $\mathbf{v}_k$ are linearly dependent. The grade r multivectors change sign, under the permutation of any two adjacent vectors

$$\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k \wedge \mathbf{v}_{k+1} \wedge \dots \wedge \mathbf{v}_r = -\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{k+1} \wedge \mathbf{v}_k \wedge \dots \wedge \mathbf{v}_r$$

and form their own vector space $\bigwedge^r \mathbb{R}^n$. It follows that the set of all r-vectors of the form

$$\mathbf{e}_{k_1} \wedge \mathbf{e}_{k_2} \wedge \dots \wedge \mathbf{e}_{k_r}, \quad \{k_1, k_2, \dots, k_r\} \ r\text{-subset of } \{1, 2, \dots, n\}$$

forms a basis of the new vector space $\bigwedge^r \mathbb{R}^n$ of dimension $\begin{pmatrix} n \\ r \end{pmatrix}$.

The full exterior algebra (or Grassmann algebra) is the vector space $\bigwedge \mathbb{R}^n$ of dimension 2^n with the r-vectors $\mathbf{e}_{k_1} \wedge \mathbf{e}_{k_2} \wedge \dots \wedge \mathbf{e}_{k_r}, 1 \le r \le n$ as a basis.

For defining the associative bilinear *geometric product* of vectors Clifford used Grassmanns multiplication rule of orthogonal vectors,

$$\mathbf{e}_k\mathbf{e}_l = -\mathbf{e}_l\mathbf{e}_k, k \neq l,$$

but set the geometric product of a vector with itself to be a real number

$$\mathbf{e}_k\mathbf{e}_k = \pm 1.$$

The associative algebra of dimension 2^n so defined it the geometric algebra $\mathbb{R}_{p,q}$. p says how many basis vectors of an orthogonal basis of the linear space R^n have positive square and the rest $q = n - p$ have negative square.

The goemetric product operating on multivectors $u, v, w \in \bigwedge \mathbb{R}^{p,q}$ can be expressed with the help of the *left contraction* $u \rfloor v \in \bigwedge \mathbb{R}^{p,q}$

(a) $\mathbf{x} \rfloor \mathbf{y} = \frac{1}{2}(\mathbf{xy} + \mathbf{yx}) \in \mathbb{R}$

(b) $\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$

(c) $(u \wedge v) \rfloor w = u \rfloor (v \rfloor w)$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,q}$ and the grade involution

$$\hat{u} = \sum_r (-1)^r \langle u \rangle_r,$$

with $\langle u \rangle_r$ the grade r-vector parts of u. (a) shows that the left contraction generalizes the usual inner product of 1-vectors. (b) indicates that the vector $\mathbf{x}$ acts like a "derivation". With the help of the left contraction, the geometric product of a vector $\mathbf{x} \in \mathbb{R}^{p,q}$ and a general multivector $u \in \bigwedge \mathbb{R}^{p,q}$ can be written as

$$\mathbf{x}u = \mathbf{x} \rfloor u + x \wedge u.$$

Demanding associativity and linearity the geometric product extends to all of $\bigwedge \mathbb{R}^{p,q}$. This gives the bilinear map which equips Grassmann's exterior algebra vector space $\bigwedge \mathbb{R}^{p,q}$ to become the geometric *algebra* $\mathbb{R}_{p,q}$.

1.4 Grade r Subspaces

All elements $u \in \mathbb{R}_{p,q}$ are defined as sums of terms of length r, summing over all $r = 0 \ldots n$. Terms of constant length r

$$\langle u \rangle_r = \sum_{1 \leq k_1 < \ldots < k_r \leq n} u_{k_1 \ldots k_r} \mathbf{e}_{k_1} \ldots \mathbf{e}_{k_r}$$

are said to be of degree (or grade) r or $r-$vectors. This notion is independent of the choice of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Linear combinations of degree $r = 2r'$ are called *even*, those with a degree $r = 2r' + 1$ are called *odd*. According to the definition, $\mathbb{R}^{p,q}$ forms the subspace of all $1-$vectors and $\mathbb{R}$ the subspace of all scalars (that is $0-$vectors or elements of grade 0). The geometric algebra $\mathbb{R}_{p,q}$ is the direct sum of its (even and odd) subspaces $\bigwedge^r \mathbb{R}^{p,q}$ of $r-$vectors:

$$\mathbb{R}_{p,q} = \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \bigwedge^2 \mathbb{R}^{p,q} \oplus \ldots \oplus \bigwedge^n \mathbb{R}^{p,q}.$$

As mentioned earlier, the dimension of each subspace $\bigwedge^r \mathbb{R}^{p,q}$ of $r-$vectors is $\begin{pmatrix} n \\ r \end{pmatrix}$. Therefore the dimension of $\bigwedge^n \mathbb{R}^{p,q}$ is $\begin{pmatrix} n \\ n \end{pmatrix} = 1$. The n-vectors are all scalar multiples of the pseudoscalar (oriented n-volume)

$$I = I_n = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n.$$

2 Geometric Algebra $\mathcal{R}_2$

2.1 Complex Numbers

$\mathbb{R}_2 = \mathbb{R}_{2,0}$ is a 4$-$dimensional real algebra with a basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2\}$. The multiplication table is

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

$\mathbb{R}_2$ has grade 0, grade 1 and grade 2 subspaces spanned by

1	$\mathbb{R}$	scalars (even)
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbb{R}^2$	vectors (odd)
$\mathbf{e}_{12}$	$\bigwedge^2 \mathbb{R}^2$	bivectors (even)

The geometric algebra $\mathbb{R}_2$ can also be written as the direct sum $\mathbb{R}_2 = \mathbb{R}_2^+ \oplus \mathbb{R}_2^-$ of its even and odd parts:
$\mathbb{R}_2^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ (even)
$\mathbb{R}_2^- = \mathbb{R}_2$ (odd).
The even part is not only a subspace but also a subalgebra isomorphic to the field $\mathbb{C}$of complex numbers.

2.2 Reflections and Rotations

In a Euclidean space, we can fix one point as origin O. All other points are then defined by their position vectors $\mathbf{x}$. Each straight line through O can be expressed in terms of a (unit length) direction vector $\mathbf{a}$ with $\mathbf{aa} = 1$. Using the fact that in the geometric product of vectors parallel components commute and orthogonal components anticommute we can reverse the sign of the orthogonal (to $\mathbf{a}$) component of a general vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{axa} = \mathbf{aa}(\mathbf{x}_\parallel - \mathbf{x}_\perp) = \mathbf{x}_\parallel - \mathbf{x}_\perp.$$

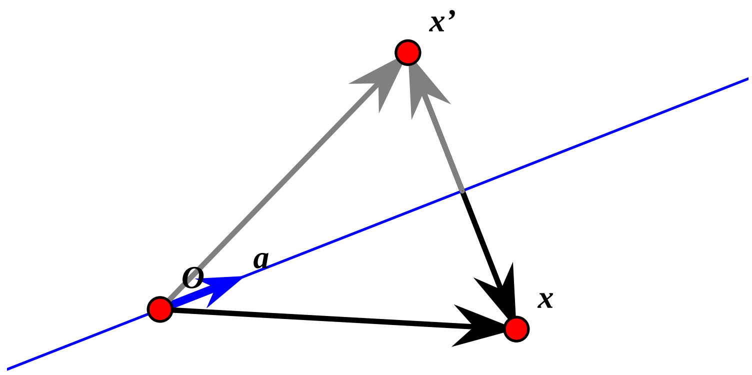

Figure 1: Reflection at a line through O in direction $\mathbf{a}$

This describes a *reflection* at the straight line through O in the direction $\mathbf{a}$. (Fig. 1.)

Two successive reflections at two lines of angle $\vartheta/2$ produce a *rotation* by the angle ϑ. If the two lines intersect in O and have unit length direction vectors $\mathbf{a}$ and $\mathbf{b}$ with angle $\vartheta/2$, the rotation by ϑ is described by (Compare figure 2.)

$$\mathbf{x}'' = \mathbf{b}\mathbf{x}'\mathbf{b} = \mathbf{baxab}.$$

The product $R = \mathbf{ba}$ is the *ro*tation opera*tor*, the *rotor*. The reverse product $\widetilde{R} = \mathbf{ab}$ is often just referred to as the *reverse*. A second successive rotation by with rotor $R' = \mathbf{cb}$ by twice the angle $\vartheta'/2$ between the vectors $\mathbf{c}$ and $\mathbf{b}$ combines as expected to give the rotation by $\vartheta + \vartheta'$

$$\mathbf{x}''' = \mathbf{cbbaxabbc} = \mathbf{caxac}$$

with rotor $R'' = R'R$. The multiplication of two rotors gives therefore a new rotor. This yields the dirotation group of rotations. (The prefix *di* indicates that the rotors $\pm R$ describe *oriented* equivalent rotations with opposite senses. Physicists call this representation also the *spin*-$\frac{1}{2}$ *representation* of the rotation group.)

2.3 Two-dimensional Point Groups

A regular polygon with k sides in two dimensions has $2k$ oriented lines of reflections, that leave the polygon invariant. This are the k lines through the k corners and the k lines through the middles of each side. (Each line is counted twice giving two orientations to each reflection, by the two orientations $\pm\mathbf{a}$ of

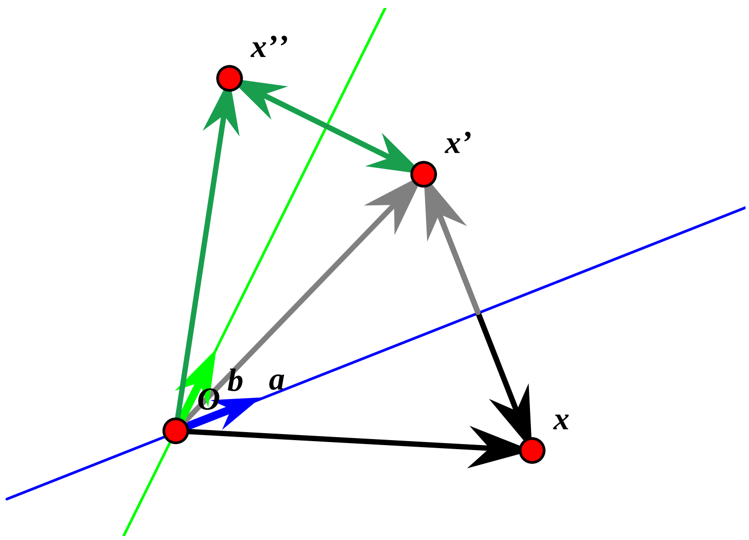

Figure 2: Rotation by two reflections.

the direction vectors.) Combining an even number of these reflections, we get the symmetry rotations of the polygon. Odd numbers of reflections simply combine to new reflections. All combinations of reflections together generate the symmetry group of the polygon, the *dihedral group* $2\mathcal{H}_k$. Compare table 1 for $k = 3$. The subgroup of symmetry rotations is simply the even subgroup $2\mathcal{C}_k$ of the dihedral group, also called *dicyclic group*. If we don't distinguish between the two "orientations" of each reflection, we have only k distinct symmetry reflections of the k-polygon, generating the group $\mathcal{H}_k$. Its even subgroup $\mathcal{C}_k$ comprises k distinct symmetry rotations.

Each dihedral group can be generated from only two reflections at straight lines passing through the center O of a k-polygon. The frist straight line passes through a corner (direction vector $\mathbf{a}$) and the second straight line through the midpoint of a side (direction vector $\mathbf{b}$). Compare figure 3 for $k = 3$. Repeated reflections at both lines generate all symmetry transformations of the dihedral group. The kth power of the elementary rotor $R = \mathbf{ba}$ leads to a full rotation by 360 degree

$$(\mathbf{ba})^k = -1, \;\; R^k \mathbf{x} \widetilde{R}^k = (-1)^2 \mathbf{x} = \mathbf{x}.$$

This relation for the kth power of R together with the unity conditions $\mathbf{a}^2 = \mathbf{b}^2 = 1$ determines the dicyclic rotor group $2\mathcal{C}_k$ completely.

3 Geometric Algebra $\mathcal{R}_3$ and Quaternions

$\mathbb{R}_3 = \mathbb{R}_{3,0}$ is an $8-$dimensional real algebra with a basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -\mathbf{i} = \mathbf{e}_{23} = \mathbf{e}_2\mathbf{e}_3, -\mathbf{j} = \mathbf{e}_{31} = \mathbf{e}_3\mathbf{e}_1, -\mathbf{k} = \mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2, i = \mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$. We have the following multiplication results:

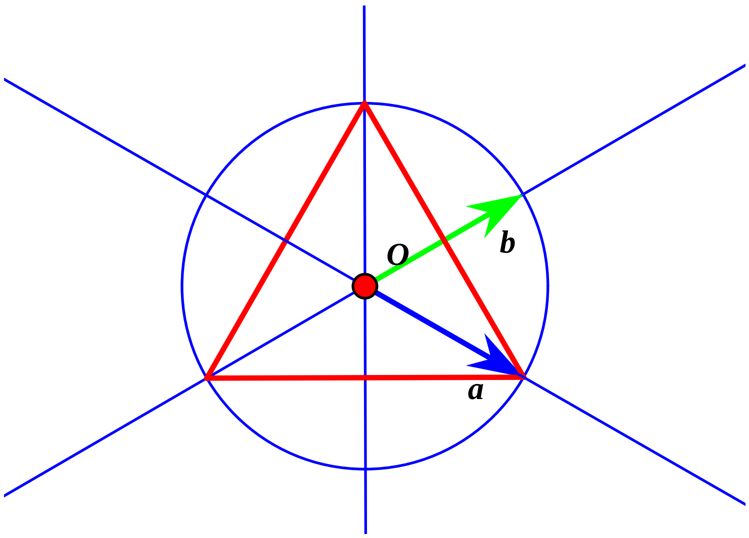

Figure 3: Vectors $\mathbf{a}, \mathbf{b}$ generate $2\mathcal{H}_3$.

3 distinct "positive sense" rotations with rotors	3 distinct "negative sense" rotations with rotors	6 distinct reflections at "oriented" lines with directions
$1 = \mathbf{a}^2 = \mathbf{b}^2$	$-1 = (\mathbf{ba})^3 = (\mathbf{ab})^3$	$\pm\mathbf{a}$
$\mathbf{ba}$	$-\mathbf{ba} = \mathbf{ba}(\mathbf{ab})^3 = (\mathbf{ab})^2$	$\pm\mathbf{b}$
$(\mathbf{ba})^2$	$-(\mathbf{ba})^2 = \mathbf{ab}$	$\pm\mathbf{bab} = \pm\mathbf{aba}$

Table 1: The 12 distinct elements of the group $2\mathcal{H}_3$.

$$1, i \text{ commute with all elements},$$
$$i\mathbf{i} = \mathbf{e}_1, i\mathbf{j} = \mathbf{e}_2, i\mathbf{k} = \mathbf{e}_3,$$
$$i^2 = \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$
$$\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}.$$

$\mathbb{R}_3$ has grade 0, grade 1, grade 2 and grade 3 subspaces spanned by

1	$\mathbb{R}$	scalars (even)
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2$	$\mathbb{R}^3$	vectors (odd)
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	$\bigwedge^2 \mathbb{R}^3$	bivectors (even)
i	$\bigwedge^3 \mathbb{R}^3$	trivectors (odd)

The geometric algebra $\mathbb{R}_3$ can also be written as the direct sum $\mathbb{R}_3 = \mathbb{R}_3^+ \oplus \mathbb{R}_3^-$ of its even and odd parts:
$\mathbb{R}_3^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3$ (even)
$\mathbb{R}_3^- = \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3$ (odd).
The even part is not only a subspace but also a subalgebra isomorphic to the algebra $\mathbb{H}$of quaternions.

4 Geometric Algebra $\mathcal{R}_{3,1}$, the Space-Time Algebra

5 Geometric Algebra $\mathcal{R}_{4,1}$ as Conformal Model of Euclidean Space

References

[1] P. Lounesto, Clifford Algebras and Spinors, CUP, Cambridge, 2001.

[2] K. Guerlebeck, W. Sproessig, Quaternionic and Clifford Calculus for Physicists and Engineers, Wiley, New York, 1997.

[3] D. Hestenes, Point Groups and Space Groups in Geometric Algebra in L. Dorst et. al. (eds.), Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Basel, 2002.

[4] A. Beutelspacher, A Survey of Grassmann's Lineale Ausdehnungslehre, in G. Schubring (ed.), Hermann Guenther Grassmann (1809-1877): Visionary Mathematician, Scientist and Neohumanist Scholar, Kluwer, Dordrecht, 1996.