The geometric product and derived products

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1 The geometric product of multivectors

The aim of this passage is to show how the geometric product of multivectors is defined in general, extending the basic geometric product of vectors given by Clifford. An alternative definition of Clifford geometric algebra, that guarantees existence as quotient algebra of the tensor algebra was given by Chevalley in 1954.[2]

In line with [4] the approach I take here is to introduce how vectors and homogeneous simple multivectors are multiplied. By linearity this extends first to the products of vectors and homogeneous multivectors and second to the products of vectors and arbitrary multivectors. Third, using the factorization of homogeneous simple multivectors into a geometric product of anticommuting vectors, we can repeatedly apply the product formula for vectors and arbitrary multivectors to calculate products of homogeneous simple multivectors and arbitrary multivectors. Fourth, by linearity this then extends to fully general geometric products of arbitrary multivectors. For brevity I always refer to a linear space of the reals $\mathbb{R}^n$ and its geometric algebra $\mathbb{R}_n$, but the formulas and definitions apply also to the geometric algebras $\mathbb{R}_{p,q}$ of linear spaces $\mathbb{R}^{p,q}$ of arbitrary signature $\{p, q\}$.

Following Grassmann's defintion, which was also adopted by Clifford, the antisymmetric outer product of two vectors $\mathbf{a}, \mathbf{b}_1 \in \mathbb{R}^n$ is given by

$$\mathbf{a} \wedge \mathbf{b}_1 = -\mathbf{b}_1 \wedge \mathbf{a},$$

mapping the pair $\mathbf{a}, \mathbf{b}_1$ to a bivector.

Following Clifford's definition, the symmetric contractions (or scalar product) of $\mathbf{a}, \mathbf{b}_1$ are given by the following grade 0 scalar

$$\mathbf{a} \rfloor \mathbf{b}_1 = \mathbf{a} \lfloor \mathbf{b}_1 = \mathbf{a} * \mathbf{b}_1 \in \mathbb{R},$$

which also corresponds to the conventional inner product of vectors.

The full geometric product of the two vectors $\mathbf{a}, \mathbf{b}_1$ is the sum

$$\mathbf{a}\mathbf{b}_1 = \mathbf{a} \rfloor \mathbf{b}_1 + \mathbf{a} \wedge \mathbf{b}_1.$$

We now define a homogeneous simple grade r multivector $B_r \in \mathbb{R}_n$ to be the product of r anticommuting vectors

$$B_r = \mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_r, \;\; \mathbf{b}_j\mathbf{b}_k = -\mathbf{b}_k\mathbf{b}_j, \;\; 1 \leq j < k \leq r \leq n.$$

The outer product of Grassmann maps $\mathbf{a}$ and B_r to a grade $r+1$ multivector which represents the grade $r+1$ part (where grade selection is indicated by angular brackets) of the geometric product $\mathbf{a}B_r$

$$\mathbf{a} \wedge B_r = \langle \mathbf{a}B_r \rangle_{r+1}.$$

The (left) contraction of a vector $\mathbf{a}$ and B_r is then given by the grade $r-1$ part of the geometric product $\mathbf{a}B_r$

$$\mathbf{a} \rfloor B_r = \langle \mathbf{a}B_r \rangle_{r-1}.$$

It can be calculated explicitely as a linear combination of homogeneous simple grade $r-1$ vectors

$$\mathbf{a} \rfloor B_r = \mathbf{a} \rfloor (\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_r) = \sum_{k=1}^{r} (-1)^{k+1} \mathbf{a} \rfloor \mathbf{b}_k (\mathbf{b}_1\mathbf{b}_2 \ldots \not{\mathbf{b}}_k \ldots \mathbf{b}_r),$$

where $\not{\mathbf{b}}_k$ means that $\mathbf{b}_k$ is to be omitted from the geometric product in round brackets. Because of the anticommutativity (i.e. orthogonality) of the vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$ we can rewrite $\mathbf{a} \rfloor B_r$ also as

$$\begin{aligned} \mathbf{a} \rfloor B_r &= \mathbf{a} \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_r) \\ &= \sum_{k=1}^{r} (-1)^{k+1} \mathbf{a} \rfloor \mathbf{b}_k (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \not{\mathbf{b}}_k \wedge \ldots \wedge \mathbf{b}_r), \end{aligned}$$

which will be of use later on.

The full geometric product $\mathbf{a}B_r$ is the sum of the (left) contraction and the outer product

$$\mathbf{a}B_r = \langle \mathbf{a}B_r \rangle_{r-1} + \langle \mathbf{a}B_r \rangle_{r+1} = \mathbf{a} \rfloor B_r + \mathbf{a} \wedge B_r.$$

Notice that we get two new grade parts: one from the left contraction resulting in a part of grade $r-1$ and one from the outer product resulting in a part of grade $r+1$.

Every multivector in a given geometric algebra can be represented as a linear combination of its homogeneous grade parts. Every grade part can be represented by a linear combination of simple multivectors of the same grade. This second summation will only become necessary for $n > 3$, because for $n \leq 3$ every homogeneous grade multivector can be factorized into a product of grade 1 vectors. For brevity I will omit the second summation over simple multivectors of the same grade.

$$B = \sum_{r=1}^{n} \langle B \rangle_r.$$

Applying linearity the product of a grade 1 vector $\mathbf{a}$ and a general multivector B is then given as

$$\mathbf{a}B = \mathbf{a}\sum_{r=1}^{n}\langle B\rangle_r = \sum_{r=1}^{n}\mathbf{a}\langle B\rangle_r = \sum_{r=1}^{n}\left(\mathbf{a}\,\rfloor\,\langle B\rangle_r + \mathbf{a}\wedge\langle B\rangle_r\right).$$

Next let us consider the geometric product of a simple grade s multivector A_s with a general multivector B. Because A_s is considered to be simple and of grade s, it can be factorized into a product of s anticommuting vectors

$$A_s = \mathbf{a}_1 \ldots \mathbf{a}_{s-1}\mathbf{a}_s, \;\; \mathbf{a}_j\mathbf{a}_k = -\mathbf{a}_k\mathbf{a}_j, \;\; 1 \leq j < k \leq s \leq n.$$

Making use of the associativity of the geometric product, we can therefore rewrite the product A_sB as

$$A_sB = \mathbf{a}_1 \ldots \mathbf{a}_{s-1}\mathbf{a}_sB = \mathbf{a}_1(\ldots(\mathbf{a}_{s-1}(\mathbf{a}_sB)\ldots).$$

This can now be explicitely calculated by repeatedly applying the previous formula for the product of a vector $\mathbf{a}$ and a general multivector B. Because each geometric multiplication of a vector $\mathbf{a}_k,\; 1 \leq k \leq s$ with a homogeneous multivector part $\langle B\rangle_r$ yields two new parts of one grade lower and one grade higher, the result of $A_s\langle B\rangle_r$ will be a linear combination of parts with grades ranging from grade $r-s$ in steps of two up to the hightest grade part $r+s$ (The left contraction will automatically be zero for the contraction of a vector with a scalar, therefore hypothetical negative grade parts will not occur, i.e. they will simply be zero.)

$$A_s\langle B\rangle_r = \langle A_s\langle B\rangle_r\rangle_{r-s} + \langle A_s\langle B\rangle_r\rangle_{r-s+2} + \ldots + \langle A_s\langle B\rangle_r\rangle_{r+s}.$$

To treat the most general case of the geometric product of two arbitrary multivectors A and B we represent A also as a linear combination of its homogeneous grade parts. (Each homogeneous grade part can in turn be represented as a linear combination of simple multivectors of the same grade. But again we omit this second summation for brevity.)

$$A = \sum_{s=1}^{n}\langle A\rangle_s.$$

The general product of two arbitrary multivectors is then by linearity

$$AB = \left(\sum_{s=1}^{n}\langle A\rangle_s\right)B = \sum_{s=1}^{n}\left(\langle A\rangle_sB\right).$$

To the expressions $\langle A\rangle_sB$ we can in turn apply factorization of the simple homogeneous grade parts of $\langle A\rangle_s$, etc. and break down the whole general product into simple elementary geometric products of grade 1 vectors (or in linear combinations of expressions just given in terms of left contractions and outer products of vectors).

To illustrate the method of explicit calculation let us conclude this section with an example. For ease of calculation we write the multivectors in terms of orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the linear space $\mathbb{R}^n$ and their geometric products. This corresponds already to decomposing the multivector factors A and B into linear combinations of homogeneous simple grade parts. The scalar coefficients for the magnitude of each homogeneous simple grade component are represented by greek letters:

$$
\begin{aligned}
A &= \alpha + \alpha_1\mathbf{e}_1 + \alpha_{12}\mathbf{e}_1\mathbf{e}_2 + \alpha_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \\
B &= \beta + \beta_2\mathbf{e}_2 + \beta_{23}\mathbf{e}_2\mathbf{e}_3 + \beta_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.
\end{aligned}
$$

The geometric product AB is

$$
\begin{aligned}
AB &= (\alpha + \alpha_1\mathbf{e}_1 + \alpha_{12}\mathbf{e}_1\mathbf{e}_2 + \alpha_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\beta + \beta_2\mathbf{e}_2 + \beta_{23}\mathbf{e}_2\mathbf{e}_3 + \beta_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \\
&= \alpha\beta + \alpha\beta_2\mathbf{e}_2 + \alpha\beta_{23}\mathbf{e}_2\mathbf{e}_3 + \alpha\beta_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + \beta\alpha_1\mathbf{e}_1 + \beta\alpha_{12}\mathbf{e}_1\mathbf{e}_2 \\
&\quad +\beta\alpha_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + \alpha_1\beta_2\mathbf{e}_1\mathbf{e}_2 + \alpha_1\beta_{23}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + \alpha_1\beta_{123}\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&\quad +\alpha_{12}\beta_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \alpha_{12}\beta_{23}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 + \alpha_{12}\beta_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&\quad +\alpha_{123}\beta_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2 + \alpha_{123}\beta_{23}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 + \alpha_{123}\beta_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.
\end{aligned}
$$

Because $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal, the square of each vector is unity $\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$ and vectors of different index anticommute, e.g. $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$, etc. We therefore have

$$
\begin{aligned}
AB &= \alpha\beta - \alpha_{123}\beta_{123}\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2 \\
&\quad +\alpha\beta_2\mathbf{e}_2 + (\alpha_1\beta + \alpha_{12}\beta_2)\mathbf{e}_1 - \alpha_{12}\beta_{123}\mathbf{e}_1^2\mathbf{e}_2^2\mathbf{e}_3 - \alpha_{123}\beta_{23}\mathbf{e}_1\mathbf{e}_2^2\mathbf{e}_3^2 \\
&\quad +(\alpha_{12}\beta + \alpha_1\beta_2)\mathbf{e}_1\mathbf{e}_2 + (\alpha\beta_{23} + \alpha_1\beta_{123})\mathbf{e}_2\mathbf{e}_3 + \alpha_{12}\beta_{23}\mathbf{e}_1\mathbf{e}_3 \\
&\quad -\alpha_{123}\beta_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\
&\quad +(\alpha\beta_{123} + \beta\alpha_{123} + \alpha_1\beta_{23})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&= \alpha\beta - \alpha_{123}\beta_{123} \\
&\quad +(\alpha_1\beta + \alpha_{12}\beta_2 - \alpha_{123}\beta_{23})\mathbf{e}_1 + \alpha\beta_2\mathbf{e}_2 - \alpha_{12}\beta_{123}\mathbf{e}_3 \\
&\quad +(\alpha_{12}\beta + \alpha_1\beta_2)\mathbf{e}_1\mathbf{e}_2 + (\alpha\beta_{23} + \alpha_1\beta_{123})\mathbf{e}_2\mathbf{e}_3 + (\alpha_{12}\beta_{23} - \alpha_{123}\beta_2)\mathbf{e}_1\mathbf{e}_3 \\
&\quad +(\alpha\beta_{123} + \beta\alpha_{123} + \alpha_1\beta_{23})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,
\end{aligned}
$$

where we have listed the result line by line in terms of grade 0 scalars, grade 1 vectors, grade 2 bivectors and grade 3 trivectors.

2 The scalar product

A useful product of multivectors $A, B \in \mathbb{R}^n$ that can be derived from their general geometric product is the socalled *scalar product* defined as the scalar part of the geometric product and indicated by an asterisk

$$A * B \equiv \langle AB \rangle_0 \in \mathbb{R}.$$

Note that the index 0 is sometimes dropped, so that angular brackets without an index come to mean the scalar part of the enclosed expression.

The scalar product of homogeneous multivectors A_s of grade s and B_r of grade r will only be different from zero, if $s = r$. Given that $s \leq r$ we can understand this restriction due to the fact that as described in the previous section, the geometric product $A_s B_r$ has as its lowest grade part a term of grade $r - s$. For the case that $s > r$ we can reverse the order of all vector factors in the geometric product AB and get

$$\langle \widetilde{A_s B_r} \rangle = \langle \tilde{B}_r \tilde{A}_s \rangle = (-1)^{s(s-1)/2}(-1)^{r(r-1)/2}\langle B_r A_s \rangle.$$

The powers of (-1) are due to the anticommutativity of the vector factors of the simple grade components of the homogeneous multivectos A_s and B_r. Now we can apply the same argument as for the case $s \leq r$ and see in general that the scalar product is only nonvanishing if $r = s$. But if $r = s$ the powers of (-1) cancel each other and we find that

$$A_s * B_r = B_r * A_s.$$

That is the scalar product of two homogeneous mulitvectors is symmetric. By linearity this extends to the scalar product of arbitrary multivectors

$$A * B = (\sum_{s=1}^{n} \langle A \rangle_s) * (\sum_{r=1}^{n} \langle B \rangle_r) = \sum_{s=1}^{n} \langle A \rangle_s * \langle B \rangle_s = \sum_{s=1}^{n} \langle B \rangle_s * \langle A \rangle_s = B * A.$$

The scalar product inherits linearity from the geometric product.

As an example let us consider the scalar product of a simple grade $s-$vector A_s with its reverse $\tilde{A}_s$

$$\tilde{A}_s * A_s = \langle \tilde{A}_s A_s \rangle = \langle \mathbf{a}_s \dots \mathbf{a}_2 \mathbf{a}_1 \mathbf{a}_1 \dots \mathbf{a}_{s-1} \mathbf{a}_s \rangle = \mathbf{a}_1^2 \mathbf{a}_2^2 \dots \mathbf{a}_s^2,$$

where we used the associativity of the geometric product, which allows us convenient pairwise multiplication of vector factors. Remember that the square of each vector is a scalar. In case that the linear space $\mathbb{R}^n$ has positive signature $(q = 0)$ the above result will be positive and by linearity we can define a positive magnitude for a general multivector A as

$$|A|^2 = \sum_{s=1}^{n} \langle \tilde{A} \rangle_s * \langle A \rangle_s \geq 0.$$

3 The outer product

The outer product in the Grassmann algebra historically preceeds the definition of geometric algebras by Clifford. But Clifford did not intend to do away with it, he rather wanted to unify Grassmann algebra and Hamilton's algebra of quaternions in a single algebraic framework. It is therefore not surprising, that the geometric product of two arbitrary multivectors comprises the outer product as the sum over the maximum grade parts of the result

$$A \wedge B = (\sum_{s=1}^{n} \langle A\rangle_s) \wedge (\sum_{r=1}^{n} \langle B\rangle_r) = \sum_{s=1}^{n}\sum_{r=1}^{n} \langle \langle A\rangle_s \langle B\rangle_r \rangle_{r+s}.$$

The outer product inherits both linearity and associativity from the geometric product.

As defined by Grassmann, the outer product of two vectors $\mathbf{a}, \mathbf{b}$ is antisymmetric

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

Calculating the reverse of two homogeneous simple grade s and grade r mulitivectors A_s and B_r we see that

$$\widetilde{A_s \wedge B_r} = \langle \widetilde{A_s B_r}\rangle_{r+s} = \langle \tilde{B}_r \tilde{A}_s\rangle_{r+s} = (-1)^{s(s-1)/2}(-1)^{r(r-1)/2}\langle B_r A_s\rangle_{r+s}.$$

On the other hand we also have

$$\langle \widetilde{A_s B_r}\rangle_{r+s} = (-1)^{(s+r)(s+r-1)/2}\langle A_s B_r\rangle_{r+s}.$$

Equating both right sides again and taking care of the powers of (-1) we end up with the symmetry formula for the outer product of homogeneous multivectors

$$A_s \wedge B_r = (-1)^{rs} B_r \wedge A_s,$$

which includes the antisymmetry of the outer product of vectors as a special case for $r = s = 1$.

3.1 The cross product of three dimensions

In conventional three-dimensional vector analysis, frequent use is made of an antisymmetric product of vectors, which results in a third vector perpendicular to the two vector factors, with the length equal to the area of the parallelogram spanned by these two vectors and with the orientation given by the socalled right hand rule. The names used for this product are: vector product, cross product or outer product and it is often indicated by an x-shaped product sign

$$\mathbf{a} \times \mathbf{b}.$$

Historically this product has actually been derived from Grassmann's outer product of vectors by mapping the resulting grade 2 bivector area element to the *dual* vector perpendicular to it. This is done by multiplication with the pseudoscalar volume element I_3.

$$\mathbf{a} \times \mathbf{b} \equiv -I_3 \mathbf{a} \wedge \mathbf{b}, \quad \mathbf{a} \wedge \mathbf{b} = I_3 \mathbf{a} \times \mathbf{b}.$$

For an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Euclidean space $\mathbb{R}^3$ we have $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Let us conclude this subsection with an example:

$$\mathbf{a} = \mathbf{e}_1 + 3\mathbf{e}_2, \;\; \mathbf{b} = \mathbf{e}_2 + 4\mathbf{e}_3,$$

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= -I_3(\mathbf{a} \wedge \mathbf{b}) \\
&= -I_3(\mathbf{e}_1 + 3\mathbf{e}_2) \wedge (\mathbf{e}_2 + 4\mathbf{e}_3) \\
&= -(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2 + 4\mathbf{e}_1\mathbf{e}_3 + 12\mathbf{e}_2\mathbf{e}_3) \\
&= -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 - 4\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3 - 12\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 \\
&= \mathbf{e}_3 - 4\mathbf{e}_2 + 12\mathbf{e}_1,
\end{aligned}$$

where we have used orthonormality of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and that $\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j$ for $k \neq j$. To recover the outer product bivector we simply mulitiply again with I_3 to get

$$\mathbf{a} \wedge \mathbf{b} = I_3(\mathbf{a} \times \mathbf{b}) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{e}_3 - 4\mathbf{e}_2 + 12\mathbf{e}_1) = \mathbf{e}_1\mathbf{e}_2 + 4\mathbf{e}_1\mathbf{e}_3 + 12\mathbf{e}_2\mathbf{e}_3.$$

It is important to note that the cross product vector construction is limitted to three dimensions. It does not work in two dimensions, because no third perpendicular dimension is available and it does not work in four and higher dimensions, because the perpendicular space is then two or higher dimensional, i.e. a unique perpendicular vector can no longer be defined. In contrast to this Grassmann's original bivector construction is not bound to the dimensionality of the space.

3.2 Linear dependence and independence

The nonzero outer product of a set of r linearly independent vectors uniquely determines an $r-$dimensional subspace of $\mathbb{R}^n$ spanned by these vectors.[4] Let us therefore proof the following proposition:

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r = 0 \iff \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^n \text{ linearly dependent.}$$

($\Leftarrow$) Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^n$ be a set of linearly dependent vectors. Without loss of generality we assume that

$$\mathbf{a}_1 = \sum_{k=2}^{r} \alpha_k \mathbf{a}_k,$$

with at least one of the coefficients $\alpha_k \neq 0$. Inserting the sum for $\mathbf{a}_1$ we get

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r = \left(\sum_{k=2}^{r} \alpha_k \mathbf{a}_k \right) \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r = 0,$$

because each term of the sum contains an expression of the form $\mathbf{a}_k \wedge \mathbf{a}_k = 0$.

($\Rightarrow$) Let the outer product of r vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^n$ be zero: $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r = 0$. But let us assume, that $A_{r-1} = \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r \neq 0$, and that the vectors $\{\mathbf{a}_2, \ldots, \mathbf{a}_r\}$ are orthogonal. Then

$$A_{r-1} = \mathbf{a}_2\mathbf{a}_3 \ldots \mathbf{a}_r$$

The following definitions will be convenient for showing that $\mathbf{a}_1$ is a linear combination of $\mathbf{a}_2, \ldots, \mathbf{a}_r$. The multiplicative inverse of each vector with respect to the geometric product is given by

$$\mathbf{a}_k^{-1} = \frac{\mathbf{a}_k}{\mathbf{a}_k^2}, \quad k = 2, \ldots, r$$

and the inverse of A_{r-1} is then given by the reversly ordered product of the inverse vectors

$$A_{r-1}^{-1} = \mathbf{a}_r^{-1} \mathbf{a}_{r-1}^{-1} \ldots \mathbf{a}_2^{-1}$$

We then have

$$A_{r-1} A_{r-1}^{-1} = \mathbf{a}_2 \ldots \mathbf{a}_{r-1}\, \mathbf{a}_r \mathbf{a}_r^{-1} \mathbf{a}_{r-1}^{-1} \ldots \mathbf{a}_2^{-1} = \mathbf{a}_2 \ldots \mathbf{a}_{r-1} \mathbf{a}_{r-1}^{-1} \ldots \mathbf{a}_2^{-1} = \ldots = 1.$$

According to our assumption, the full geometric product of $\mathbf{a}_1$ and A_{r-1} is

$$\mathbf{a}_1 A_{r-1} = \mathbf{a}_1 \rfloor A_{r-1} + \mathbf{a}_1 \wedge A_{r-1} = \mathbf{a}_1 \rfloor A_{r-1}$$

and therefore

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{a}_1 A_{r-1}\, A_{r-1}^{-1} = (\mathbf{a}_1 \rfloor A_{r-1}) A_{r-1}^{-1} \\ &= \sum_{k=2}^{r} (-1)^{r-k} (\mathbf{a}_1 \rfloor \mathbf{a}_k) \mathbf{a}_r^{-1} \mathbf{a}_{r-1}^{-1} \ldots \not{\mathbf{a}}_k \ldots \mathbf{a}_2^{-1} \mathbf{a}_2 \ldots \mathbf{a}_k \ldots \mathbf{a}_r \\ &= \sum_{k=2}^{r} (\mathbf{a}_1 \rfloor \mathbf{a}_k) \mathbf{a}_r^{-1} \mathbf{a}_{r-1}^{-1} \ldots \not{\mathbf{a}}_k \ldots \mathbf{a}_2^{-1} \mathbf{a}_2 \ldots \not{\mathbf{a}}_k \ldots \mathbf{a}_r \mathbf{a}_k \\ &= \sum_{k=2}^{r} (\mathbf{a}_1 \rfloor \mathbf{a}_k) \mathbf{a}_k. \end{aligned}$$

End of the proof.

A lemma of the above proposition is, that

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r \neq 0 \iff \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^n \text{ linearly independent.}$$

Another lemma is that for a set of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^n$ we have for $\mathbf{a} \in \mathbb{R}^n$:

$$\mathbf{a} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\} \iff \mathbf{a} \wedge \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r = 0.$$

Up to an arbitrary nonzero scalar factor, each homogeneous simple multivector is therefore in one to one correspondence with a subspace of $\mathbb{R}^n$. This is the reason, why the operations of join, intersection (meet) and projection on subspaces can be easily be expressed by geometric products of corresponding homogeneous simple multivectors. The pseudoscalar I_n of a geometric algebra $\mathbb{R}_n$ corresponds to the space $\mathbb{R}^n$ itself.

4 Right and left contraction

We have already encountered the left and right contractions of two grade 1 vectors and the left contraction of a vector and a homogeneous grade r multivector. But the left and right contractions can be generalized to apply to arbitrary multivectors.

The most general approach is to define left and right contractions solely in terms of the outer product and the scalar product. In this way the *left contraction* can be defined as[5]

$$C * (A \rfloor B) \equiv (C \wedge A) * B, \qquad \forall A, B, C \in \mathbb{R}_n,$$

and the *right contraction* as

$$(B \lfloor A) * C \equiv B * (A \wedge C), \qquad \forall A, B, C \in \mathbb{R}_n.$$

Note that both left and right contractions are linear, because the scalar product and the outer product used in these definitions are linear. Decomposing the multivectors A, B, C grade by grade we get

$$A = \sum_{k=1}^{n} A_k, \;\; B = \sum_{l} B_l, \;\; C = \sum_{m} C_m.$$

Inserting this into the expression for the left conctraction we get by linearity of both the outer and the scalar product

$$\begin{aligned}
(C \wedge A) * B &= \sum_{k,l,m} (C_m \wedge A_k) * B_l = \sum_{m,k} \langle C_m A_k B_{l=m+k} \rangle_0 \\
&= \sum_{m,k} \langle C_m \langle A_k B_{m+k} \rangle_m \rangle_0 = \sum_m C_m * \left(\sum_k \langle A_k B_{m+k} \rangle_m \right) \\
&= C * (A \rfloor B).
\end{aligned}$$

This gives an explicit expression for the left contraction in terms of the grade parts of A and B

$$A \rfloor B = \sum_{k,l} \langle \langle A \rangle_k \langle B \rangle_l \rangle_{l-k}.$$

In analogy to this we get for the right contraction

$$B \lfloor A = \sum_{k,l} \langle \langle B \rangle_l \langle A \rangle_k \rangle_{l-k}.$$

Note that in both cases $m = l - k \geq 0$, i.e. combinations of l and k with $l < k$ do not contribute.

It is now straightforward to see that the contractions of two homogeneous multivectors of the same grade give their scalar product

$$\langle A \rangle_k \rfloor \langle B \rangle_k = \langle A \rangle_k \lfloor \langle B \rangle_k = \langle \langle A \rangle_k \langle B \rangle_k \rangle_{k-k=0} = \langle A \rangle_k * \langle B \rangle_k,$$

and that, e.g. the left contraction of a vector $\mathbf{a}$ with a homogeneous grade r multivector B_r will give a homogeneous grade $r-1$ multivector

$$\mathbf{a} \rfloor B_r = \langle \mathbf{a} B_r \rangle_{r-1}.$$

In order to derive convenient explicit formulas for the calculation of the contractions of homogeneous simple multivectors, we will show the following formula:

$$A_r \rfloor (B_s \rfloor C_t) = (A_r \wedge B_s) \rfloor C_t,$$

where A_r, B_s and C_t are supposed to be homogeneous simple multivectors of grade r, s and t, respectively. For the left side to be nonzero, we must have $t \geq s$ and $t - s > r$ which is equivalent to $t > r + s$. To perform the proof, we scalar multiply with an arbitrary multivector $D \in \mathbb{R}_n$ from the left to get by repeated application of the defining relationship of the left contraction

$$\begin{aligned} D * [A_r \rfloor (B_s \rfloor C_t)] &= (D \wedge A_r) * (B_s \rfloor C_t) = (D \wedge A_r \wedge B_s) * C_t \\ &= [D \wedge (A_r \wedge B_s)] * C_t = D * [(A_r \wedge B_s) \rfloor C_t], \\ &\quad \forall D \in \mathbb{R}_n. \end{aligned}$$

And hence $A_r \rfloor (B_s \rfloor C_t) = (A_r \wedge B_s) \rfloor C_t$. Note that for conducting the proof no conditions on the values of the grades r, s and t had to be made. The left contraction "takes care" of this.

Let us now look in detail at the left contraction of two homogeneous simple multivectors A_r and B_s of grades r and s, respectively, given by the following vector factorizations

$$A_r = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r, \;\; B_s = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_s$$

With repeated application of the formula, which we have just proved, we can rewrite the left contraction of A_r and B_s as

$$\begin{aligned} A_r \rfloor B_s &= (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r) \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_s) \\ &= ((\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{r-1}) \wedge \mathbf{a}_r) \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_s) \\ &= (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{r-1}) \rfloor (\mathbf{a}_r \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_s)) \\ &= \ldots = \\ &= \mathbf{a}_1 \rfloor (\mathbf{a}_2 \rfloor (\ldots \rfloor (\mathbf{a}_r \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_s)) \ldots)). \end{aligned}$$

By repeated application of a previously shown formula

$$\mathbf{a} \rfloor B_s = \sum_{k=1}^{r} (-1)^{k+1} \mathbf{a} \rfloor \mathbf{b}_k (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \not{\mathbf{b}}_k \wedge \ldots \wedge \mathbf{b}_s),$$

and by reordering we finally reach a very explicit formula for $A_r \rfloor B_s$

$$A_r \rfloor B_s = \sum_{j_1 < \ldots < j_r} \epsilon(j_1 \ldots j_s) A_r \rfloor (\mathbf{b}_{j_1} \wedge \ldots \wedge \mathbf{b}_{j_r})(\mathbf{b}_{j_{r+1}} \wedge \ldots \wedge \mathbf{b}_{j_s}).$$

Note that the left contraction is only nonzero for $r \leq s$. For $r > s$ the right side has to be replaced by zero. Each $j_k \leq s$ is a positive integer with $j_1 < \ldots < j_r$ and with $j_{r+1} < \ldots < j_s$. $\epsilon(j_1 \ldots j_s) = 1$ for even permutations of $(1, 2, \ldots, s)$ and -1 for odd permutations, respectively. Compare also formula (1.40) on page 11 of [4].

Reordering and the explicit formula for $\mathbf{a} \rfloor B_s$ further yield for $r = s$ that

$$\tilde{A}_r * B_r = \tilde{A}_r \rfloor B_r = \sum_{k=1}^{r} (-1)^{j+k} (\mathbf{a}_j \rfloor \mathbf{b}_k)(\mathbf{a}_r \wedge \ldots \check{\mathbf{a}}_j \ldots \wedge \mathbf{a}_1) \rfloor (\mathbf{b}_1 \wedge \ldots \check{\mathbf{b}}_k \ldots \wedge \mathbf{b}_r).$$

Continuing this expansion for the homogeneous simple grade $r - 1$ multivectors $(\mathbf{a}_r \wedge \ldots \check{\mathbf{a}}_j \ldots \wedge \mathbf{a}_1)$ and $(\mathbf{b}_1 \wedge \ldots \check{\mathbf{b}}_k \ldots \wedge \mathbf{b}_r)$ we get the expansion formula for the determinant of the matrix with coefficients $f_{jk} = \mathbf{a}_j \rfloor \mathbf{b}_k = \mathbf{a}_j * \mathbf{b}_k$. This is also precisely the reason why the determinant definition in geometric algebra given previously as

$$\det(f) = \underline{f}(I_n) * \tilde{I}_n = \underline{f}(\tilde{I}_n) * I_n.$$

agrees with the traditional matrix calculus definition. We just need to set $r = n$, $B_r = I_n$ and $A_r = \underline{f}(I_n) \Leftrightarrow \tilde{A}_r = \underline{f}(\tilde{I}_n)$.

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