E. Hitzer, *Foundations of Multidimensional Wavelet Theory: The Quaternion Fourier Transform and its Generalizations,* Preprints of Meeting of the Japan Society for Industrial and Applied Mathematics, ISSN: 1345-3378, Tsukuba University, 16-18 Sep. 2006, Tsukuba, Japan, pp. 66,67.

Foundations of Multidimensional Wavelet Theory:

The Quaternion Fourier Transform and its Generalizations

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Keywords: Multidimensional Wavelets, Quaternion Fourier Transform, Clifford geometric algebra

1. Basic facts about Quaternions

Gauss, Rodrigues and Hamilton's 4D quaternion algebra H over R:

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,\ i^2 = j^2 = k^2 = ijk = -1, \qquad (1)$$

with isomorphisms $H \approx Cl(0,2) \approx Cl^+(3,0)$. $Cl^+(3,0)$ is the even subalgebra of Clifford geometric algebra Cl(3,0), with basis $\{1,\ e_{32} = e_3e_2,\ e_{13} = e_1e_3,\ e_{21} = e_2e_1\}$ for an orthonormal basis $\{e_1, e_2, e_3\}$ of R^3. The quaternion

$$q = q_r + q_i i + q_j j + q_k k \in H, \quad q_r, q_i, q_j, q_k \in R \qquad (2)$$

has the *quaternion conjugate* (reversion in $Cl^+(3,0)$)

$$\tilde{q} = q_r - q_i i - q_j j - q_k k\ , \qquad (3)$$

This leads to the *norm* of $q \in H$

$$||q|| = \sqrt{(\tilde{q}q)} = \sqrt{(q_r^2 + q_i^2 + q_j^2 + q_k^2)}\ . \qquad (4)$$

Quaternions (and quaternion valued functions) can be split in two ways:

$$q = q_r + iq_i + q_j j + iq_k j \ \text{ or } \ q = q_+ + q_- = (q+iqj)/2 + (q-iqj)/2\ . \qquad (5)$$

The second split allows to write

$$q_\pm = \{q_r \pm q_k + i(q_i \mp q_j)\}(1 \pm k)/2 = (1 \pm k)\{ q_r \pm q_k + j(q_j \mp q_i)\}/2\ . \qquad (6)$$

Applying (5) and (6) to the quaternionic kernel *K* = exp(-ixu) exp(-jyv) gives

$$K_\pm = \exp(-i(xu \mp yv))\ (1 \pm k)/2 = (1 \pm k)\ \exp(-j(yv \mp xu))\ /2\ . \qquad (7)$$

For 2D quaternion valued functions *f,g* we can define the inner product ($\mathbf{x} = xe_1 + ye_2$)

$$(f,g) = \int f(\mathbf{x})\tilde{g}(\mathbf{x})\ dxdy\ , \qquad (8)$$

with real scalar part

$$\langle f,g \rangle = \int \langle f(\mathbf{x})\tilde{g}(\mathbf{x}) \rangle\ dxdy\ , \qquad (9)$$

and norm

$$||f|| = \sqrt{(f,f)} = \sqrt{\langle f,f \rangle}\ . \qquad (10)$$

2. Quaternion Fourier Transform (QFT)

Ell [1] defined the QFT for application to 2D linear time-invariant systems of PDEs. Later it was extensively applied to 2D image processing [2], including color. This spurred research into optimized numerical applications. The invertible QFT of a 2D quaternion valued signal f is defined as

$$\mathrm{F}\{f\} = \int \exp(-\mathrm{i}xu)\, f(\mathbf{x})\, \exp(-\mathrm{j}yv)\, dxdy. \tag{11}$$

The scalar product (9) gives the Plancherel theorem

$$\langle f,g\rangle = \langle \mathrm{F}\{f\}, \mathrm{F}\{g\}\rangle/(2\pi)^2 . \tag{12}$$

As corollary we get the Parseval (Rayleigh's) theorem for signal energy preservation

$$\| f \| = \| \mathrm{F}\{f\} \| / 2\pi . \tag{13}$$

Useful for solving PDEs with polynomial coefficients are the following moment formulas ($\mathbf{u}= ue_1+ve_2$)

$$\mathrm{F}\{x^m y^n f\}(\mathbf{u}) = \mathrm{i}^m\, d^{m+n}\, \mathrm{F}\{f\}(\mathbf{u}) /(du^m dv^n)\, \mathrm{j}^n , \tag{14}$$

and

$$\mathrm{F}\{\mathrm{i}^m f \mathrm{j}^n\}(\mathbf{u}) = \mathrm{i}^m\, \mathrm{F}\{f\}(\mathbf{u})\, \mathrm{j}^n . \tag{15}$$

Equations (5) and (15) reduce the computation of $\mathrm{F}\{f\}$ to the four QFTs of real functions f_r , f_i , f_j , f_k . And (15) shows that every theorem for the QFT of real 2D functions results in a theorem for quaternion-valued functions. For example a general linear non-singular transformation A of the QFT of 2D real signals can in this way be generalized to 2D quaternion-valued functions (for B compare [2])

$$\mathrm{F}\{f(A\mathbf{x})\}(\mathbf{u}) = |\det B|/2\, [\mathrm{F}\{f\}(B_+\mathbf{u}) + \mathrm{F}\{f\}(B_-\mathbf{u}) + \mathrm{i}\, (\mathrm{F}\{f\}(B_+\mathbf{u}) + \mathrm{F}\{f\}(B_-\mathbf{u}))\, \mathrm{j}] . \tag{16}$$

Instead of (11) we can define the invertible right sided QFT (Clifford FT) as

$$\mathrm{F}_r\{f\}(\mathbf{u}) = \int f(\mathbf{x})\, \exp(-\mathrm{i}xu)\, \exp(-\mathrm{j}yv)\, dxdy , \tag{17}$$

and obtain the Plancherel theorem

$$(f,g) = (\mathrm{F}_r\{f\}, \mathrm{F}_r\{g\})/(2\pi)^2 . \tag{18}$$

As corollary we again get a Parseval identity

$$\| f \| = \| \mathrm{F}\{f\} \| / 2\pi = \| \mathrm{F}_r\{f\} \| / 2\pi . \tag{19}$$

For F_r linearity and dilation properties hold, some other properties need commutation dependent modifications.

3. $GL(\mathrm{R}^2)$ Transformation Properties

We observe that the split (7) results in two complex kernels $K_\pm$ with complex units i (or j) apart from $(1\pm \mathrm{k})/2$. We therefore analyze the transformation properties of $\mathrm{F}\{f\}$ in terms of $\mathrm{F}\{f_\pm\}$. We can prove that

$$\mathrm{F}\{f_\pm\}(\mathbf{u}) = \int f_\pm \exp(-\mathrm{j}(yv\mp xu))\, dxdy = \int \exp(-\mathrm{i}(xu\mp yv))\, f_\pm\, dxdy . \tag{20}$$

Every $A \in GL(\mathrm{R}^2)$ can be decomposed to $A=TR=RS,$ with R a rotation, T and S symmetric with positive and negative eigenvalues (ev.). Positive (negative) ev. correspond to stretches (reflections and stretches perpendicular to line of reflection). Rotations can be composed by two reflections $R_{ab}=U_aU_b$. Elementary transformations are hence reflections (Cartan) and stretches. In Clifford geometric algebra U_n is given by the vector $\mathbf{n}$ normal to the line of reflection $U_n\mathbf{x} = -\mathbf{n}^{-1}\mathbf{x}\mathbf{n}$. Using $xu+yv = \mathbf{x}\cdot\mathbf{u}$, $-xu+yv = \mathbf{x}\cdot(U_{e1}\mathbf{u})$ we get

$$\mathrm{F}\{f_-\}(\mathbf{u}) = \int f_- \exp(-\mathrm{j}\, \mathbf{x}\cdot\mathbf{u})\, dxdy , \qquad \mathrm{F}\{f_+\}(\mathbf{u}) = \int f_+ \exp(-\mathrm{j}\, \mathbf{x}\cdot(U_{e1}\mathbf{u}))\, dxdy . \tag{21}$$

We therefore get for automorphisms $A \in GL(\mathrm{R}^2)$, $A^{\dagger -1}$ the adjoint inverse transformation of A

$$F\{f_-(A\mathbf{x})\}(\mathbf{u}) = |\det A^{-1}|\, F\{f_-\}(A^{\dagger -1}\mathbf{u}) ,\quad F\{f_+(A\mathbf{x})\}(\mathbf{u}) = |\det A^{-1}|\, F\{f_+\}(U_{e1}A^{\dagger -1}U_{e1}\mathbf{u}) . \tag{22}$$

The combination of (22) gives therefore

$$F\{f(A\mathbf{x})\}(\mathbf{u}) = |\det A^{-1}|\, [\, F\{f_-\}(A^{\dagger -1}\mathbf{u}) + F\{f_+\}(U_{e1}A^{\dagger -1}U_{e1}\mathbf{u})\,] . \tag{23}$$

For axial stretches we get ($ab \neq 0$, $a,b \in R$)

$$F\{f(A_s\mathbf{x})\}(\mathbf{u}) = F\{f\}(ue_1/a+ve_2/b)/|ab| . \tag{24}$$

For reflections we get ($\mathbf{a}' = U_{e1}\mathbf{a}$)

$$F\{f(U_a\mathbf{x})\}(\mathbf{u}) = F\{f_-\}(U_a\mathbf{u}) + F\{f_+\}(U_{a'}\mathbf{u}) . \tag{25}$$

For rotations we get

$$F\{f(R\mathbf{x})\}(\mathbf{u}) = F\{f_-\}(R^{-1}\mathbf{u}) + F\{f_+\}(R\mathbf{u}) . \tag{26}$$

4. Generalization to spatio-temporal signals

Quaternion isomorphisms and $GL(R^{n,m})$ transformation laws allow generalization to higher dimensions. As an example we take an isomorphism to a subalgebra of the spacetime [3] algebra Cl(3,1) with time vector e_0, 3D volume $I_3 = e_1e_2e_3$ and spacetime volume $I_4 = e_0e_1e_2e_3$, all three with negative square. $\{e_0, I_3, I_4\}$ generate an algebra isomorphic to quaternions.
This leads to an invertible spacetime FT for 4D multivector valued Cl(3,1) functions f

$$F_{st}\{f\}(\mathbf{u}) = \int \exp(-e_0ts)\, f(\mathbf{x})\, \exp(-I_3\mathbf{x'}\cdot\mathbf{u'})\, d^4x, \tag{27}$$

With $d^4x = dtdxdydz$, $\mathbf{x} = te_0+\mathbf{x}'$, $\mathbf{x}'=xe_1+ye_2+ze_3$, $\mathbf{u}= se_0+\mathbf{u}'$, $\mathbf{u}'=ue_1+ve_2+we_3$. The space time split

$$f_\pm = (f \pm e_0 f I_3)/2 \tag{28}$$

yields therefore the transformation formulas (comp. [4,5])

$$F_{st}\{f_\pm\}(\mathbf{u}) = \int f_\pm(\mathbf{x})\, \exp(-I_3(\mathbf{x'}\cdot\mathbf{u'} \mp ts))\, d^4x = \int \exp(-e_0(ts \mp \mathbf{x'}\cdot\mathbf{u'}))\, f_\pm(\mathbf{x})\, d^4x . \tag{29}$$

Our new results will serve for the further development of discrete and continuous multivector wavelets.

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[5] For further literature see mathematics publication section of: http://sinai.mech.fukui-u.ac.jp/