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Review text:

The treatise suits graduate students and researchers in mathematics and quantum physics.

First an axiomatic introduction to Clifford algebras $Cl(p, q = n - p)$ is given. Casanova generalizes Frenet's formulas to $R(n, 0)$, calculates squares of a canonical n-vectors $A_n = e_1 e_2 \dots e_n$ and considers subspace r-vectors and their commutativity.

He defines inversions of power $k \in \mathbb{R} : Xx = k$, $(OM = x, OM' = X)$. This is extended to differential curves $X(t), x(t)$, and compositions of inversions. Casanova studies the resulting maps and sets of invariant points by solving Clifford algebraic equations.

$u = x + \varepsilon y,\ \varepsilon^2 = 1,\ \varepsilon \in Cl(p, q),\ x, y \in \mathbb{R}$ is a *hyperbolic* number with conjugate $\bar{u} = x - \varepsilon y$. $\bar{v}u = k \in \mathbb{R}$ is a *hyperbolic* inversion. He goes on to study the hyperbolic inverse images of certain conics, and curves invariant under hyperbolic inversion (example: invariant cubic). He also investigates compositions, and shows that hyperbolic inversions are isogonal transformations. Real and complex quaternions are briefly set in context.

Next the inverse of $A, A' \in Cl(p, q),\ A^2 = \alpha + i\beta,\ {A'}^2 = \alpha' + \varepsilon\beta',\ i^{-1} = -i,\ \varepsilon^{-1} = \varepsilon,\ \alpha, \beta, \alpha', \beta' \in \mathbb{R}$ are investigated and extended to the inverse of the gradient operator $D = e^{\mu}\partial_{\mu}$. Applications in $Cl(1, 3)$ with orthonormal vector basis $\{e_0, e_1, e_2, e_3\}$, are Hestenes' solution to the Maxwell equation $DF = 0$, with bivector $F = f \exp(ik \cdot x)$, f constant bivector, k vector, and the Hestenes Dirac equation (HDE) $D\psi = m\psi e_0 e_1 e_2$. [D.O. Hestenes, *Space-time Algebra*, Gordon&Breach, 1966]

The following section deals with hyperbolic roots of equations of degree n applied to straight line with plane curve intersections and linear differential equations. Then Casanova investigates bivector products. He defines *hyperbolic* and *parabolic analytical* functions. The latter are power series in $v = x + \alpha y,\ x, y \in \mathbb{R},\ \alpha \in Cl(p,q),\ \alpha^2 = 0$, including log and $[f(v)]^{g(v)}$.

The prelude for the physical applications is the decomposition of $Cl(1,3)$ bivectors $B = \exp(-i\varphi)b$, b bivector with $b^2 > 0,\ \varphi \in \mathbb{R}$, i pseudo-scalar ($i^2 = -1$). Casanova defines Dirac wave functions as even $Cl(1,3)$ multivectors, and Lorentz (rotor) transformations $x' = Rx\tilde{R}$ with $R = \bar{R}, R\tilde{R} = 1$ ($\tilde{R}$ reverse, $\bar{R}$ grade involution.)

He now derives the canonical $Cl(1,3)$ rotor decomposition $R = \exp(i\varphi b')\exp(\vartheta b')$ with bivector $b' : {b'}^2 = 1,\ \varphi, \vartheta \in \mathbb{R}$. (For zero square bivector part $\langle R\rangle_2 : R = \pm\exp\langle R\rangle_2$.) This leads to the *trigonometric form of the wave function* (TFWF) $\psi = (\rho \exp i\beta)^{1/2}R,\ 0 \leq \rho \in \mathbb{R},\ \beta \in \mathbb{R}$ (Yvon-Takabayashi angle). The Dirac vector derivative D in $Cl(1,3)$ unifies the Maxwell equations to $DF = J$, F field bivector, J current vector, A vector potential ($F = D \wedge A$) and Lorentz force vector $F \cdot J$. The gauge invariant HDE (charge e) becomes $D\psi = (m\psi e_o + eA\psi)e_1 e_2$.

The gauge invariant, self-dual field quantities are $\psi g \tilde{\psi}$ with $g = 1, i, e_0, e_1 e_2$, and for the spin vector $J_3,\ g = e_3$, respectively. Utilizing the TFWF, HDE solutions times $e_1 e_0$ represent positrons. Regarding β, Casanova derives from the HDE an equation for the electron under the Lorentz force. Neglecting $\frac{\hbar^2}{c}$-order terms it describes the relativistic dynamics of an electron with proper mass $m_0 \cos\beta$ $(0 \leq \beta \leq \pi)$, $\cos\beta$ distinguishing positive and negative proper energies. The divergence of the spin vector is $D \cdot J_3 = 2m\rho \sin\beta$.

Next Casanova gives a brief (HDE based) discussion of the virtual vs. objective nature of Louis de Broglie's formulas. In his further discussion of HDE solutions, he considers plane waves (also in constant magnetic field), positron observability, negative potential energy states, interference, electron localization, Coulomb energy eigenstates, probability, kinetic moments, ground state energies, spherical solutions, subquantum determinism, hidden variables, and wave vs. particle nature.

Casanova extends the electron HDE and derives nucleon (protons, neutrons, Xi-baryons) TFWFs in the presence of pion fields. The nucleon TFWF rotates e_3 into the isospin vector I_3. Isospin divergence, flux conservation, charge-baryon-isospin relations and (rotating $e_1 e_2$) the magnetic moment are calculated. Regarding nuclear forces by virtual photon and pion exchange, Casanova makes wide-ranging philosophical remarks on *subquantum* physics, mathematics and even biology.