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Review text:

This paper is of interest to graduate students and researchers in hypercomplex algebras (including Clifford geometric algebras) and non-Euclidean differential geometry.

The authors study the geometry generated by *commutative Segre quaternions* (CSQ). After the definition and classification of CSQ into *elliptic*, parabolic and hyperbolic, the treatment focuses on canonical elliptic CSQ. The product of a CSQ with its three principal conjugations motivates the decomposition into two complex variables. Interpreting the four components of a CSQ as 4D point coordinates, distance (symmetric form of degree four) preserving transformations (roto-translations) can be defined.

A non-flat quaternion space is thus introduced by defining its line element as a fourth degree differential form. In analogy to Gauss' study of surfaces by means of complex numbers, *conformal coordinates* for CSQ are introduced for constant curvature CSQ geometries. Using the calculus of variations, and introducing a Christoffel-like symbol, Euler's equation for geodesics is derived.

Next the roto-transformations are introduced as CSQ Moebius transformations (linear-fractional mappings), keeping a certain factorized form of the CSQ line element invariant.

The rest of the paper is devoted to deriving explicit geodesic equations for positive constant curvature canonical elliptic CSQ spaces, involving only exponential and trigonometric functions of the CSQ coordinates. To achieve this the authors adapt the integration method developed by *Beltrami*. In the first step they apply the first order Beltrami differential parameter to each factor of the CSQ line element, giving rise to two geodesic arc lengths. In the second step the two CSQ line element factors are written in terms of the two geodesic arc lengths. Then three equations for the CSQ geodesics are derived in terms of the two arc lengths. Re-parametrization leads to the final explicit coordinate equations for the CSQ geodesics.

The authors remark that extension of their method to the other CSQ classes (parabolic and hyperbolic) is straight-forward.