This is a review text file submitted electronically to MR.

Reviewer: Hitzer, Eckhard, M. S.

Reviewer number: 036883

Address:

Department of Physical Engineering, University of Fukui, 3-9-1 Bunkyo, Fukui, Fukui 910-8507, JAPAN
hitzer@mech.fukui-u.ac.jp

Author: Hestenes, David; Holt, Jeremy W.

Short title: Crystallographic space groups in geometric algebra.

MR Number: 2297981

Primary classification: 20H15

Secondary classification(s): 82D25, 15A66, 20F55, 20F05

Review text:

This article is suitable for crystallographers interested in the geometrical foundations of crystallographic space groups, and in general for everyone who wants to take a geometrically motivated approach to crystal symmetry, but avoid coordinate based matrix representations.

First necessary basics of *Clifford geometric algebra* (GA) are introduced. A vector a normal to a plane of reflection represents this reflection by $x \to -axa$. In GA products of vectors (*versors*) thus represent compositions of reflections. This leads to Lipschitz (Clifford) groups representing the orthogonal transformations of non-Euclidean vector spaces ^{p,q} in $Cl_{p,q}$, also called *versor groups* by the authors.

This is already sufficient to present the 10 (32) point groups in two (three) dimensions by specifying the angles between two (three) vectors by one (two) angular indices $\pi/p, \pi/q$. This gives rise to a compact point group notation, isomorphic to table 7 in [H. Coxeter, W. Moser, Generators and Relations for Discrete Groups, Springer, NY, 1980].

Then follows a brief introduction to *conformal geometric algebra* in $Cl_{4,1}$, also known as the *Horosphere* model of Euclidean space ^{3} with extra null-directions for origin and infinity. The conformal group $C(4,1)$ is isomorphic to the orthogonal group $O(4,1)$, which has the Euclidean group $E(3)$ as its subgroup, leaving the point at infinity invariant. Crystallographic space groups are discrete subgroups of $E(3)$. The translations of $E(3)$ become therefore also versors in conformal GA.

Two figures (Figs. 5 and 6) specify the lattice vectors in two (three) dimensions according to crystal system and Bravais lattice. These lattice vectors generate both the point group versors and the lattice translation versors. This is already sufficient to denote all symmorphic space groups in two and three dimensions by compact symbols combining the Bravais letter with the GA point group symbol.

For the non-symmorphic groups combinations with fractional translations to form glide reflections and screw displacements have to be specified. This is done by adding indices before and after the angular index for two dimensional space groups (before, between and after the two indices for three dimensional space groups). These new GA space group symbols can be used alternatively to international Herrmann-Maugin symbols. Together with the two figures, the GA space group symbols allow to write down the complete set of versor generators for each space group (tables 4 and 5).

Finally a brief comparison with other notations is made and the flexibility to chose equivalent sets of versor generators is explained. Conformal GA naturally encodes orientation of geometric objects, which opens the way to similarly encode even larger classes of space groups. Another benefit is the increasing use of conformal GA in computer graphics leading to the space group visualization project mentioned in ref. 19 [www.spacegroup.info].