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Review text:

This paper is of interest to graduate students and researchers of theoretical physics, who are interested in the algebraic structure of field equations.

The paper deals with commutative quaternions, distinct from Hamilton's non-commutative quaternions. Apart from commutative elliptic (Segre) quaternions, commutative parabolic, and in particular commutative hyperbolic quaternions are also treated. The reader is advised to exercise care in keeping these cases apart.

Section 2 reviews the formulation of two-dimensional electrostatic field equations with complex variables. This allows to unify inhomogeneous (real part) and homogeneous equations (imaginary part) in one complex equation. The homogeneous (non-metric) part represents an integrability condition, the inhomogeneous (metric) part gives rise to Poisson's equation. Using instead hyperbolic variables (as in two-dimensional special relativity) leads to an inhomogeneous part describing wave propagation.

Related to section 2, appendix A reviews the inhomogeneous Cauchy-Riemann system and its solution. [Remark: Different sign conventions have been used in equations (62), (63), and (64)!]

Section 3 summarizes the algebraic properties of commutative quaternions and the properties of holomorphic functions of quaternions. As for the algebra, the multiplication table, characteristic matrix, conjugations, norms, (characteristic) planes of zero divisors, and idempotent basis are treated. As for holomorphic functions of quaternions a generalized set of Cauchy-Riemann (GCR) conditions is established. Moreover the differential properties of quaternion functions subject to the GCR conditions and written in the idempotent basis are investigated.

Section 4 examines commutative elliptic quaternion fields, which are shown to be equivalent to two two-dimensional (2D) (complex) harmonic fields. Elliptic and hyperbolic quaternion fields are shown to have the same integrability conditions, but different inhomogeneous (metric dependent) field equations: two independent 2D Poisson equations (elliptic quaternions) and two 2D wave equations (hyperbolic quaternions).

Appendix B reviews relevant parts of Cauchy's theory of partial differential systems: initial value problem, characteristic surfaces (with physical interpretation), bicharacteristics (rays), and wave-particle duality (relation of characteristic varieties and bicharacteristics) [T. Levi-Civita, *Characteristics of Differential Systems and Wave Propagation*, Zanichelli, Bologna (1931).] Section 4 applies this to show how the characteristic surfaces of quaternion field equations can be related to forward and backward propagating wave fronts. This treatment emphasizes the relationship between the Lie groups of quaternion functions and the symmetries of the associated quaternion geometry.

Finally section 5 compares Maxwell's electromagnetic field equations with quaternion field equations. The resulting differences may perhaps be reduced in the future by examining higher dimensional commutative quaternions. It may also be interesting to draw a comparison with the unified Maxwell equation in the Clifford geometric algebra of 4D Minkowsky space-time, where the Dirac derivative is also invertible as an integral operator, directly expressing the fields in terms of the sources [D. Hestenes, *Space-Time Algebra*, Gordon and Breach, NY (1966)], and where holomorphic (analytic) functions are generalized to monogenic functions [C. Doran, A. Lasenby, *Geometric Algebra for Physicists*, CUP, Cambridge (2003)].