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Review text:

This textbook addresses graduate students and researchers interested in geometric algebras like Grassmann algebra, Clifford algebra and related algebras, as well as automated (geometric) theorem proving, and conformal models of classical geometries.

The book begins with a foreword by D. Hestenes. The introduction starts with Leibniz dream of a geometric calculus dealing directly with geometric objects, whose realization for classical geometries is seen in (affine) Grassmann-Cayley (GC) algebra, Clifford (geometric) algebra, and conformal geometric algebra. It defines the notion of hierarchical advanced invariants of clear geometric meaning, and with certain symmetry properties. Such advanced invariants are excellent for multiplicative representation of constructions; a bracket-oriented representation, elimination, expansion for factored and shortest result (breefs) method has been developed, and Clifford factorization is introduced. This combination allows in most cases monomial or binomial expression analytic proofs of geometric theorems.

Chapter 2 introduces the concepts of projective space, bracket algebra and GC algebra. It begins with classical invariant theory of homogeneous polynomials of brackets (determinants of n atomic vectors in V^n). Their relations are syzygies, e.g. Grassmann-Plücker or van der Waerden syzygies. Sweedler's notation of set partition and summation is explained. The direct use of brackets in representation and computation avoids middle expression swell. The nD bracket algebra is the commutative ring generated by brackets of any n-tuples of $m(> n)$ vectors. For $m < n$ socalled deficit brackets are introduced, representing e.g. classical covariants. A Grassmann algebra viewed as vector space (Grassmann space) is the space of all antisymmetric tensors. A Grassmann space with outer and meet product is called Grassmann-Cayley (GC) algebra. In the context of Grassmann coalgebra, the new concept of total meet product is defined. The next section is devoted to Cayley expansion, the procedure of eliminating all meet products, so that the result contains only brackets and outer products of vectors, it is seen as a procedure of algebraization. Here and in Appendix A Caley expansion theory for 2D and 3D projective geometries is studied in detail. The next section is devoted to Grassmann factorization. Caley factorization translates algebra back to geometry and is inverse to Caley expansion.

Chapter 3 develops projective incidence geometry with Cayley bracket algebra, which has superior symbolic manipulation performance in enumerative geometry. The rich structure of the Cayley bracket algebra approach includes covariants, basic, rational and advanced invariants, and allows with its algebraic manipulations vast improvements over traditional approaches of biquadratic final polynomials or the area method. The chapter goes on to develop factorization, contraction and division techniques for bracket algebra. Then rational invariants, rational monomials composed of invariant ratios and rational bracket monomials, are studied. All this is applied in automated theorem proving where a Cayley expression conclusion, as solution of multivector equations of projective incidence geometry must be verified. Important are the construction (batch) sequence and the (batch) elimination sequence with dynamic order. Then a first automatic theorem proving algorithm in projective incidence geometry is introduced which yields a proving procedure and nondegeneracy conditions. Several examples are considered explicitly including the still open problem of Erdös consistent 5-tuples.

Chapter 4 treats projective conic geometry with bracket algebra and with quadratic GC algebra. Fundamental is a degree-4 bracket polynomial for six points being on the same conic (Pascal's theorem). In this context GC bracket algebra for conics determined by points, by polars and tangents, and for conic intersections is developed. Bracket-oriented representation aims at optimizing the algebraic representations of all points in a bracket at the same time for producing a factored and shortest polynomial result. A number of simplification techniques like (pseudo-) conic transformation, conic contraction, and factorization techniques in coconic computing are discussed, alongside a wide range of explicit examples. This leads to an algorithm for hand-checkable machine proofs in projective conic computing. The limitations of conic geometry with bracket algebra lead to the approach of enlarging the projective vector space V^n to the space of (step two) symmetric tensors generated in V^n. The quadratic GC algebra and bracket algebra upon this space give an intrinsic language for computing conic geometric problems. This is closely related to the approach of

C. Perwass in [Geometric Algebra with Applications in Engineering, Springer, NY, 2009.]. Extensions to cubic GC algebra and *m-ic* GC algebra and the respective bracket algebras for degree-m algebraic curves are possible.

Chapter 5 first introduces inner-product Grassmann algebra and its bracket algebra as the algebras for orthogonal geometry, and for its invariants under the orthogonal group, respectively. Aiming for a complete system of advanced orthogonal invariants naturally leads to Clifford algebra. Translation from Clifford algebra to inner-product Grassmann algebra is Clifford expansion, back translation is called Clifford factorization. Note that the choice of H. Li's definition of the non-zero inner product of a scalar and a multivector scales the multivector. Inner product algebra includes the outer and inner product, as well as the dual operator w.r.t. a fixed unit pseudocalar; and naturally generates a GC algebra, capable of doing all projective geometric computing.

Regarding advanced invariants graded inner-product bracket algebra is introduced for computations with cosines of angles of highdimensional linear objects. Completion of the system is reached in Clifford algebra based on the (almost invertible) geometric (Clifford) product, which includes (completes) the inner and outer products. Thus Clifford algebra unifies and contains all previously discussed algebras. Clifford algebra is the unique associative and multilinear algebra for the extension of isometries. The geometric product contains all geometric relations between two geometric objects represented by invertible Clifford algebra elements. Traditional representations include hypercomplex numbers (Hamilton, Clifford), multilinear tensors, martices (Cartan), geometric algebra (D. Hestenes) and Clifford deformations of Grassmann algebra. Versors (products of invertible vectors) represent all orthongonal transformations in V^n via their graded adjoint action (Cartan-Dieudonné). In the remainder Clifford expansion theory is treated in detail.

Chapter 6 is devoted to geometric algebra as the version of Clifford algebra preferring the use of multiplication to addition. The great value of the geometric product is its associativity and the commutation symmetries within the grading operator. Further major techniques developed are ungrading, monomial simplification, rational Clifford expansion and factorization. These techniques allow the transitions from graded Clifford polynomials to basic multivectors (expansion) or to Clifford polynomials (ungrading). Next the technique of versor compression (minimizing the number of effective vectors in a versor) is studied in detail from a symbolic computation point of view, including a section on obstructions to versor compression (parabolic and hyperbolic rotors). Following the earlier investigation of GC coalgebra (chp. 2) now Clifford coalgebra, Clifford summation and factorization is studied, categorized by the number of participating monomials and the types of $\mathbb{Z}$-grading operators. Finally it is shown how Clifford bracket algebra extends graded inner-product algebra to an algebra of higher level invariants. To reduce middle expression swell, the homogeneous model of Euclidean geometry is needed.

Chapter 7 develops the conformal model of Euclidean geometry [F.L. Wachter (1792-1817), S. Lie (PhD 1872), etc.], which effectively represents the (affine) translation structure and is fully compatible with the inner product structure. Now circles and spheres become primitive objects like points, lines and (hyper) planes. Conformal GC algebra is the algebra of intersections and extensions of these objects, including nonlinear geometric objects of constant curvature. Regarding homogeneous and cartesian coordinates affine GC algebra is discussed, the moment-direction representation of affine subspaces, the affine boundary operator, and the cartesian model of Euclidean space.

The conformal model of Euclidean geometry in $\mathbb{R}^{n+1,1}$ is essentially based on Lie sphere geometry, but discards the orientation dimension. The inner product of two points is the Euclidean distance, a basic invariant. The embedding of points via a nonlinear isometry seems complicated, but leads to astonishing simplifications for Euclidean geometric computing. Changing the origin-infinity Minkowski plane of the model induces conformal (Möbius) transformations in $\mathbb{E}^n$, and orthogonal transformations in $\mathbb{R}^{n+1,1}$ induce conformal transformations in $\mathbb{E}^n \cup \{e_\infty\}$. The geometric meaning and the inner products of vectors of different signature are discussed in detail. The homogeneous models of nD Euclidean geometry is the set of null vectors $\mathcal{N}$ in $\mathbb{R}^{n+1,1}$ and a fixed $e_\infty \in \mathcal{N}$. Salient features are origin-freeness, and dilation invariant representations of spheres and planes. Positive-vector representations of (pencils of) spheres and hyperplanes and a general discriminant of three vectors allows to completely classify the relations of three objects (points, spheres, or planes).

Switching to the dimension independent dual *Minkowski blade* representation permits to extend the Grassmann structure of projective incidence geometry to Euclidean incidence geometry including notions of collinearity, cocircularity, parallelism, perpendicularity and tangency of Euclidean points, lines, circles, planes and spheres of various dimensions. This leads to GC algebra of $\mathbb{R}^{n+1,1}$ with inner product and boundary operator, as conformal GC algebra over $\mathbb{E}^n$, and the rich geometry of Minkowski blades is investigated in detail. A highlight is the application of the total meet of chp. 2 to two Minkowski blades representing two circles. The next section is devoted to the Lie model of (oriented) spheres and hyperplanes in $\mathbb{R}^{n+1,2}$, as well as contact geometry. Within this framework the Apollonian contact problem of constructing a circle tangent to 3 objects (lines, points or circles) is easily solved and completely classified and can be extended to nD.

Chapter 8 is on conformal Clifford algebra (GC algebra and Clifford algebra on the conformal model) which is seen to provide a unified framework including projective, affine, Euclidean, hyperbolic, spherical, elliptic and conformal geometries and the generalization of dual vector algebra of 3D affine objects to nD. In the geometry of positive monomials (positive constituent vectors) versors for conformal transformations, and geometric products of Minkowski blades are discussed in detail. Then the Cayley transform and the exterior exponential are studied using rotors in the conformal model represented via their Lie algebra generators. Then the classical work of Vahlen (1992) on representing elements of $Cl(n+1,1)$ with $Cl(n)$-matrices is introduced, leading to the nD extension of fractional linear transformations as Möbius transformations in $\mathbb{E}^n$. Then affine geometry with dual Clifford algebra is discussed, but seen to depend strongly on the origin, a restriction overcome in the homogeneous model. For the rest the conformal (and homogeneous) models of spherical and hyperbolic geometry are established, and their ultimate unification with Euclidean geometry in the unified algebraid framework for classical geometries is introduced. Here a single algebraic identity can be translated into different geometric theorems in different geometries. Even in the same geometry a single algebraic identity may have various geometric explanations, which can be exploited. As an example Simson's theorem is discussed. The universal conformal model allows to describe a classical geometry either via a conformal distance function or via conformal frames in conformal coordinates.

Throughout the book features a host of worked out examples and gives a very unified systematic presentation of a wide variety of material.

Comments to the MR Editors: Sorry for taking so long.